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Jump-Diffusions with Controlled Jumps: Existence and Numerical Methods

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A comprehensive development of effective numerical methods for stochastic control problems in continuous time, for reflected jump-diffusion models, is given in earlier work by the author. While these methods cover the bulk of models which have been of interest to date, they do not explicitly deal with the case where the jump itself is controlled in the sense that the value of the control just before the jump affects the distribution of the jump. We do not deal explicitly with the numerical algorithms but develop some of the concepts which are needed to provide the background which is necessary to extend the proofs of earlier work to this case. A critical issue is that of closure, i.e., defining the model such that any sequence of (systems, controls) has a convergent subsequence of the same type. One needs to introduce an extension of the Poisson measure (which serves a purpose analogous to that served by relaxed controls), which we call the relaxed Poisson measure, analogously to the use of the martingale measure concept given earlier to deal with controlled variance. The existence of an optimal control is a consequence of the development. © 2000 Academic Press

1. INTRODUCTION

A comprehensive development of effective numerical methods for stochastic control problems in continuous time is given in [10, 11, 16]. These methods cover the bulk of models which have been of interest to date. The process models include controlled diffusions and reflected diffusions both with and without jumps (the jumps being defined by a Poisson measure driving process). The jump terms were not controlled however. Recent applications to telecommunications systems have involved

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controlled jumps, and an example from a polling problem where the polled queues are occasionally unavailable will be given. The paper will outline the extension of the results of [16] to this problem. One place where such problems arise is in wireless communications where the sources are sending data which are created in a random and bursty way, buffered until transmitted, and the sources can be occasionally unavailable to the base antenna due to their physical movement. In this case, the state will be the total amount of work that is in the buffers of the sources and the control policy is the balance of the total buffered work between the sources. The jump is due to the increase in work if one source becomes unavailable for a period of time which is longer than what is needed to handle all of the work in the other available source plus what arrives during that interval. More detail is given below. We will not be concerned with numerical algorithms per se, but only with providing crucial background which allows the extension of the results in [16].

Let $\{\mathcal{F}_t,\ t \geq 0\}$ be a filtration on some probability space and $\mathcal{U}$ a compact set in some Euclidean space. Let $w(\cdot)$ be a standard $\mathcal{F}_t$-Wiener process, $N(\cdot)$ an $\mathcal{F}_t$-Poisson measure, and $u(\cdot)$ (the control) an $\mathcal{F}_{t-}$-adapted and $\mathcal{U}$-valued process. More will be said about it later. The jump rate of $N(\cdot)$ is $\lambda < \infty$ and the jump distribution is $\Pi(\cdot)$, where the jumps are confined to a compact set Γ. Let $G \subset \mathbb{R}^r$, Euclidean r-space, be a hyperrectangle with faces ∂G_i, $i = 1, \ldots, 2r$, and a nonempty interior. Let n_i denote the (unit length) interior normals to ∂G_i and let d_i be unit vectors such that $\langle n_i, d_i \rangle > 0$.

Consider the controlled reflected jump-diffusion process defined by

$$\begin{aligned} dx(t) &= b(x(t), u(t))\,dt + \sigma(x(t))\,dw(t) \\ &\qquad + \int_\Gamma q(x(t-), \gamma, u(t)) N(dt\,d\gamma) + dz(t), \qquad x(t) \in G. \quad (1.1) \end{aligned}$$

The process is constrained to the set G by the reflection process $z(\cdot)$. The direction of reflection on the interior of the boundary face ∂G_i is d_i, and the direction on an edge or corner is allowed to be any convex combination of the directions on the adjoining faces. Conditions on d_i and on the functions in (1.1) will be specified later. Write $z(t) = \sum_i d_i y_i(t)$, where $y_i(\cdot)$ can increase only when $x(t) \in \partial G_i$. Under condition (A2.1), $y(\cdot) = \{y_i(\cdot)\}$ is uniquely determined by $z(\cdot)$.

Uncontrolled (and unreflected) processes of this type are discussed in [6–8] and many other places concerned with the general theory of stochastic differential equations. An introduction to the problem with reflections, also known as the *Skorohod problem* is in [3, 16, 15]. Alternatively to (1.1), the process can be described in terms of the interjump sections and the state (or state and control) dependent jump rate and state and control dependent jump distribution, without explicitly introducing the Poisson measure and function $q(\cdot)$. Let $\beta > 0$ and $c_i \geq 0$ and let $k(\cdot)$ be continuous and real valued. We will work with the discounted cost function

$$W_\beta(x,u) = E\int_0^\infty e^{-\beta t}\left[k(x(t),u(t))\,dt + c'\,dy(t)\right]. \tag{1.2}$$

Constrained models such as (1.1) in compact state spaces arise as heavy traffic limits to controlled queueing and communications networks [1, 15, 17]. Additionally, whatever the original state space of the problem, for numerical purposes it often needs to be reduced to some compact set. See Fig. 1 for a two-dimensional illustration of G and the d_i.

An Example of (1.1): *Controlled Polling.* Examples of (1.1) have arisen in recent applications to telecommunications as heavy traffic limits of controlled networks. Consider the following problem of scheduling a single server to serve two competing queues, each of which receives data in a random way. The connections between the sources and the server are subject to breakdown. For example, let there be two mobile sources and let the server be a fixed antenna, with the sources moving to inaccessible places from time to time. When one source becomes inaccessible, only the accessible one can be served. Since the server would have a capacity that is greater than that needed to handle the mean load of any one source, if the breakdown period is large relative to the queue size of the available source at the time of breakdown, then the server will have a lot of idle time until the other source becomes available again. This leads to a jump in the total workload. Clearly the jump depends on the queue sizes just before breakdown. The control is over the service policy when both sources are available, hence over the jump size. In the heavy traffic limit, where time and amplitude are scaled such that the period of unavailability condenses to a point, equations such as (1.1) arise. See [1] for a complete treatment.

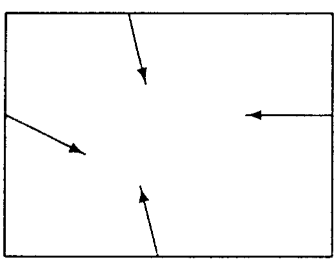

FIG. 1. An example of the state space and reflection directions.

The specific model can be described in the following way. For $i = 1,2$, let $\tau^s_{i,k}$, $k < \infty$, be mutually independent random variables which are exponentially distributed with rate $\bar{\lambda}^s_i$ for $\tau^s_{i,k}$. Define $\nu_{i,l} = \sum_{k=1}^{l} \tau^s_{i,k}$. Let $\tau^v_{i,l}$, $k < \infty$, $i = 1,2$, be mutually independent random variables with bounded variances, the distribution depending on i only, and independent of the $\tau^s_{i,l}$. Let $x(t)$ represent the (scaled in time and value) total workload. This evolves as a diffusion between breakdowns, where the Wiener process is independent of the τ variables. The jumps due to the breakdowns of the connection to source i occur at times $\nu_{i,l}$, $l < \infty$, and have values $\xi^n_{i,l}$ defined by

$$\begin{aligned} \xi^v_{1,l} &= \left[\rho_1 \tau^v_{1,l} - \left(x(\nu_{1,l}-) - u(\nu_{1,l}) \right) \right]^+, \\ \xi^v_{2,l} &= \left[\rho_2 \tau^v_{2,l} - u(\nu_{1,l}) \right]^+, \end{aligned} \tag{1.3}$$

where $0 \le u(t) \le x(t)$, $\rho_i > 0$, and $\sum_i \rho_i = 1$. The parameter ρ_i is the mean fraction of the capacity of the channel which is used by source i. These formulas determine the $q(\cdot)$ and $N(\cdot)$ in (1.1), with $\lambda = \sum_i \bar{\lambda}^s_i$. This example is just one dimensional, since it is expressed in terms of the total queued workload, but there are analogous multidimensional problems. In (1.1), the control was constrained to lie in a compact set $\mathscr{U}$. But in the example the constraint was $0 \le u \le x$. Since the state x is bounded, the constraint $0 \le u \le x$ can be inserted by replacing u by $\max\{u, x\}$ in the dynamics and cost function. We assume, without loss of generality, that the functions have been defined so that the jumps will not take $x(\cdot)$ out of G.

The control of the jump as in the form (1.1) has not been treated to date. The actual numerical algorithms which would be used are obvious adaptations of those in [11, 16] with the control added to the jump term. The aim of this paper is to provide the background needed to adapt the proofs of convergence in [16] to models such as (1.1). The issue of convergence arises from the fact that if $u^n(\cdot)$ is a sequence of admissible controls with corresponding solutions $x^n(\cdot)$, then there might not be a (weakly) convergent subsequence of $\{x^n(\cdot), u^n(\cdot)\}$ whose limit satisfies (1.1) for some Wiener process, Poisson measure, and admissible control. This issue of closure arises even without the jump term, and in that case has led to the notion of relaxed control [16, 18]. If the variance term $\sigma(\cdot)$ were also subject to control, then the relaxed control is extended into the so-called martingale measure [11]. Indeed, the motivation behind the martingale measure extension in [11] can be used to handle the controlled jump terms. In a sense the "relaxed Poisson measure" which will be introduced is a natural analog of the martingale measure process in [11]. The conditions which will be used are stronger than necessary, but they allow us to illustrate the ideas without excessive technical encumbrance.

The idea of relaxed control is classical. But, owing to its importance a few words will be said about it in Section 2, where it is introduced to provide motivation and background for the controlled jump case of the next section. This concept involves an extension of the classical concept of control and has considerable theoretical advantages. While the space of allowed controls is enlarged, the problem has not really changed and has the same infimum of the costs. The most important property is that of "closure," in the sense that any sequence of (systems, controls) has a convergent subsequence with a limit of the same type. The concept of relaxed Poisson measure, developed in Section 4, carries the relaxed control concept to the controlled Poisson measure case. Approximation theorems, that show that any control in the extended sense can be approximated by an ordinary control, are also given. The state equation and Bellman equation for the controlled polling example is given in Section 5. This is not needed for the sequel, but it illustrates the type of PDE that appears in such problems, at least formally. The details of the numerical algorithms are as in [16]. But Section 5 shows how to extend part of the convergence proofs of that reference to the present case.

2. BACKGROUND

Relaxed Deterministic Controls. Consider the unconstrained ODE $\dot{x} = b(x, u)$, where the control function $u(t)$ takes values in a compact set $\mathcal{U}$. For this motivating discussion, let us use the discounted cost function

$$\int_0^\infty e^{-\beta t} k(x(t), u(t))\, dt,$$

where $b(\cdot)$ and $k(\cdot)$ are bounded and continuous. Define an admissible ordinary control to be a measurable $\mathcal{U}$-valued function. It is well known [5, 16, 18] that there is not always an optimal in the class of admissible ordinary controls. One needs to enlarge the class by introducing the so-called relaxed controls [5, 16, 18]. This enlarged class is used for theoretical purposes only.

An admissible relaxed control $m(\cdot)$ is a measure on the Borel sets of $\mathcal{U} \times [0, \infty)$ such that $m(\mathcal{U} \times [0, t]) = t$. There is a derivative $m_t(\cdot)$ such that for any bounded Borel set B in $\mathcal{U} \times [0, \infty)$,

$$m(B) = \int_0^\infty \int_{\mathcal{U}} I_{\{(\alpha, t) \in B\}} m_t(d\alpha)\, dt,$$

and $m_t(A)$ can be assumed to be measurable for each Borel $A \subset \mathscr{U}$. One can define

$$m_t(A) = \lim_{\delta \to 0} [m(t, A) - m(t - \delta, A)]/\delta. \tag{2.1}$$

An ordinary control $u(\cdot)$ can be represented in terms of a relaxed control $m(\cdot)$, where $m_t(A) = I_A(u(t))$, where $I_A(u)$ is unity if $u \in A$ and is zero otherwise. Rewrite the ODE in relaxed control notation as

$$\dot{x}(t) = \int_{\mathscr{U}} b(x(t), \alpha) m_t(d\alpha). \tag{2.2}$$

The weak topology [16] will be used on the space of admissible relaxed controls. There will be an optimal control in the class of admissible relaxed controls. The inf of the values of the cost function over the admissible ordinary controls equals that over the admissible relaxed controls, and any admissible relaxed control can be arbitrarily well approximated by an admissible ordinary control.

For a concrete illustration, consider the example where $u^n(t) = \alpha_1$ on the intervals $[2k/n, 2k/n + 1/n)$, $k = 0, 1, \ldots,$ and equals α_2 on the alternate intervals. Let $m^n(\cdot)$ denote the relaxed control representation of $u^n(\cdot)$. Then $m^n(\cdot)$ converges to the admissible relaxed control $m(\cdot)$ where $m_t(\cdot)$ is concentrated on the points α_i, each with mass $1/2$. The dynamics in the limit ODE (2.2) are just the *averaged* dynamics. The use of $m(\cdot)$ in (2.2) is not a randomization. It is an actual *averaging* of the dynamics. For the controlled jump problem, the relaxed control will still average the drift, but its effect on the jump will be that of a *randomization.* Relaxed controls are a mathematical convenience and are often indispensable in proving existence of optimal controls, convergence of numerical methods, and approximation results in control problems in general. They are not for practical use and they do not appear in the numerical algorithms.

Stochastic Differential Equations and Controlled Drift. Let $\mathscr{B}(S)$ be the σ-algebra over the Borel sets in the metric space S. The predictable σ-algebra $\mathscr{F}^p$ is defined as the minimal σ-algebra over the sets in $\mathscr{F}_\infty \times \mathscr{B}([0, \infty))$ which contains the sets $(s, t] \times A$, where $A \in \mathscr{F}_s$. A predictable process (also called an $\mathscr{F}_t$-predictable process) is measurable on $\mathscr{F}^p$. Consider (1.1) with $q(\cdot)$ *not* depending on the control. Given the filtration $\{\mathscr{F}_t,\ t \geq 0\}$, the *stochastic relaxed control* $m(\cdot)$ is said to be *admissible* if $m(A \times [0, \cdot\,])$ is an $\mathscr{F}_t$-adapted process for all $A \in \mathscr{B}(\mathscr{U})$, and for almost all ω, $m(\cdot)$ is an admissible deterministic relaxed control. The derivative $m_t(A)$ defined in (2.1) exists for almost all (ω, t) and is $\mathscr{F}_t$-predictable. We also say that the triple $(w(\cdot), N(\cdot), m(\cdot))$ is *admissible*. Rewrite (1.1) as (2.3), but *without* the control in the jump term,

$$dx(t) = \int_{\mathscr{U}} b(x(t), \alpha) m_t(d\alpha)\, dt + \sigma(x(t))\, dw(t) + \int_{\Gamma} q(x(t-), \gamma) N(d\gamma\, dt) + dz(t), \qquad x(t) \in G. \quad (2.3)$$

One could add a control in the diffusion term [11, 13, 14], but we wish to avoid the substantially more complicated notation.

In relaxed control notation, the cost function is

$$W_\beta(x, u) = E\int_0^\infty \int_{\mathscr{U}} e^{-\beta t}\left[k(x(t), \alpha) m_t(d\alpha)\, dt + c'\, dy(t)\right]. \quad (2.4)$$

Assumptions. (A.2.1) Fix a corner of G, and let i index the adjoining faces. Define v_{ij} by $1 - v_{ii} = \langle n_i, d_i \rangle$, and for $i \neq j$ set $v_{ji} = |\langle n_i, d_j \rangle|$. Then the spectral radius of the matrix $V = \{v_{ij};\, i, j\}$ is less than unity. This must hold for each corner.

(A.2.2) $b(\cdot)$, $\sigma(\cdot)$, $q(\cdot)$, $k(\cdot)$ are continuous.

(A.2.3) For each initial condition and admissible triple $(w(\cdot), N(\cdot), m(\cdot))$, there is a weak sense solution to the system (2.3) without the jumps, and it is unique in the weak sense.

Let τ denote the time of the first jump of

$$p(t) = \int_0^t \int_\Gamma \gamma N(ds\, d\gamma). \quad (2.5)$$

The solution to (1.1) or (2.3) on $[0, \tau)$, whether or not the jump time and value are controlled, does not depend on the value of the jump. Then the distribution of the first jump is well defined. One can proceed in this way to define the solution for all t for either (1.1) or (2.3). From this point of view, where the role of both $q(\cdot)$ and $N(\cdot)$ is suppressed, we will need the additional condition that the distribution of the jump is weakly continuous in the state and control values.

The methods to be used are based on the theory of weak convergence [2, 4] and extend the ideas in [11, 16]. The path space for the process $x(\cdot)$, $z(\cdot)$ is $D(\mathbb{R}^r; 0, \infty)$, the set of $\mathbb{R}^r$-valued functions on $[0, \infty)$ which are right continuous and with the Skorohod topology used. The tightness criterion to be used implicitly is Theorem 2.7b of [9].

3. THE JUMP CONTROL PROBLEM: FORMULATION

The notion of relaxed control needs to be extended to deal with the problem form (1.1). In the absence of controlled jumps, the concept of relaxed control was introduced to deal with the problem of closure, the fact that the limit of a sequence of processes $x^n(\cdot)$ with ordinary controls used was not necessarily representable as a process of the type (2.3) subject to an ordinary control. However, if $x^n(\cdot)$ is the solution driven by the admissible triple $(w^n(\cdot), N^n(\cdot), m^n(\cdot))$ (where the distribution of $(w^n(\cdot), N^n(\cdot))$ does not depend on n), then $(x^n(\cdot), m^n(\cdot), w^n(\cdot), N^n(\cdot))$ has a weakly convergent subsequence whose limit $(x(\cdot), m(\cdot), w(\cdot), N(\cdot))$ solves (2.3) and $(w(\cdot), N(\cdot), m(\cdot))$ is admissible.

Furthermore, any relaxed control can be approximated by an ordinary (always admissible) control in the sense that the costs and the distribution of the process under the relaxed control can be arbitrarily well approximated by those under some ordinary control. Thus, the introduction of relaxed controls does not change the problem, but simplified the analysis. The numerical methods always give a control in feedback form, and the relaxed control concept is used only for the proofs. An analogous extension is needed to handle the jump control problem. The extension and motivation will be developed in steps.

A Motivating Example. The following example will illustrate the underlying issue of "closure" and guide us to the solution. Suppose that the admissible control $u(t)$ takes the two values α_i, $i = 1, 2$. Divide time into intervals of length $\delta > 0$, and divide each of these into subintervals of lengths $v_1\delta, v_2\delta$, where $v_1 + v_2 = 1$. Use the control value α_1 on $[k\delta, k\delta + v_1\delta)$ and use α_2 on $[k\delta + v_1\delta, k\delta + \delta)$, $k = 0, 1, \dots$. Let $x^\delta(\cdot)$ (resp., $u^\delta(\cdot)$) denote the associated solution (control, resp.) to (1.1). Let $I_i^\delta(s)$ denote the indicator function of the event that α_i is used at time s. Then the jump term in (1.1) takes the form

$$
\begin{aligned}
J^\delta(t) &= \int_0^t \int_\Gamma q\big(x^\delta(s-), \gamma, u^\delta(s)\big) N(ds\, d\gamma) \\
&= \sum_i \int_0^t \int_\Gamma I_i^\delta(s) q\big(x^\delta(s-), \gamma, \alpha_i\big) N(ds\, d\gamma). \qquad (3.1)
\end{aligned}
$$

Let $m^\delta(\cdot)$ denote the relaxed control representation of $u^\delta(\cdot)$. Then $m_t^\delta(\alpha_i) = m_t^\delta(\{\alpha_i\}) = I_i^\delta(t)$. Let $\delta \to 0$. Then $m^\delta(\cdot)$ converges weakly to $m(\cdot)$ with $m_t(\alpha_i) = v_i$. The set (over δ and the jump index) of all jumps is tight as is the set of interjump sections of $x^\delta(\cdot)$. Fix a weakly convergence subsequence of the interjump sections and jumps. Then (between jumps) the limit of the chosen subsequence can be represented as

$$dx(t) = \int_{\mathscr{U}} b(x(t), \alpha) m_t(d\alpha)\, dt + \sigma(x(t))\, dw(t). \tag{3.2}$$

The limit of $J^{\delta}(\cdot)$ along the chosen subsequence can be expressed in the form

$$\sum_i \int_0^t \int_\Gamma q(x(s-), \gamma, \alpha_i) \overline{N}_i(ds\, d\gamma), \tag{3.3}$$

where $\overline{N}_i(\cdot)$, $i = 1, 2$, are mutually independent Poisson measures with jump distributions $\Pi(\cdot)$ and jump rates $v_i \lambda$. The limit triple $(w(\cdot), \overline{N}_i(\cdot),$ $= 1, 2, m(\cdot))$ is admissible. The form (3.3) emphasizes the fact that the control value which affects the jump is the result of a randomization. This type of approximation and weak convergence analysis could be carried out for any number of values of the control. It can also be adapted to the case where the fractions of the intervals on which the α_i are used are time dependent in a nonanticipative way. For example, let $m^{\delta}(\cdot)$ denote the relaxed control representation of an $\mathscr{F}_t$-predictable process $u^{\delta}(\cdot)$ which takes only a finite number of values. Let $x^{\delta}(\cdot)$ denote the associated solution and let $(x^{\delta}(\cdot), m^{\delta}(\cdot))$ converge weakly to $(x(\cdot), m(\cdot))$. Let $\{\mathscr{F}_t, t \geq 0\}$ denote the filtration which this process induces (perhaps augmented by a Wiener process and Poisson measure which are independent of the other processes). Then there is a standard $\mathscr{F}_t$-Wiener process $w(\cdot)$ and $\mathscr{F}_t$-adapted counting measure valued processes $\overline{N}_i(\cdot)$ such that the set solves (3.2) between jumps and the jumps are represented by (3.3), but where the former jump rate $v_i \lambda$ of $\overline{N}_i(\cdot)$ is replaced by the random and time varying (always $\mathscr{F}_t$-predictable) quantity $\lambda m_t(\alpha_i)$. Thus, the limit of the jump term can be represented in terms of a set of (extended) Poisson measures with jump rates depending on the derivative of the limit relaxed control. The $\overline{N}_i(\cdot)$ would not be independent, but the martingales defined by

$$\begin{aligned}&\int_0^t \int_\Gamma I_i^{\delta}(s) q(x^{\delta}(s-), \gamma, \alpha_i) N(ds\, d\gamma)\\ &\qquad - \lambda \int_0^t \int_\Gamma q(x^{\delta}(s-), \gamma, \alpha_i) \Pi(d\gamma) m_s^{\delta}(\alpha_i)\, ds\end{aligned} \tag{3.4a}$$

converge weakly to the processes

$$\int_0^t \int_\Gamma q(x(s-), \gamma, \alpha_i) \overline{N}_i(ds\, d\gamma) - \lambda \int_0^t \int_\Gamma q(x(s-), \gamma, \alpha_i) \Pi(d\gamma) m_s(\alpha_i)\, ds \tag{3.4b}$$

which are $\mathscr{F}_t$-martingales.

There is an alternative representation of (3.3) which is sometimes useful. Extend the Poisson measure as follows. Let $\Pi_0(\cdot)$ be Lebesgue measure on $[0, 1]$, and let $\overline{N}(ds\, d\gamma\, d\gamma_0)$ denote the Poisson measure with jump rate λ and jump distribution $\Pi(d\gamma)\Pi_0(d\gamma_0)$. Let the control take a finite number of values $\{\alpha_i,\ i \le p\}$, and define $\mu_0(t) = 0$ and $\mu_i(t) = \sum_{j=1}^{i} m_t(\alpha_i)$. Then write the ith summand in (3.3) in the form

$$\int_0^t \int_\Gamma \int_0^1 I_{\{\gamma_0 \in (\mu_{i-1}(t),\, \mu_i(t)]\}} q(x(s-), \gamma, \alpha_i) \overline{N}(ds\, d\gamma\, d\gamma_0), \tag{3.5}$$

The representation (3.5) yields a process (interjump and jump) with the same probability distribution. The form of (3.5) emphasizes, again, that the actual realization of the jump value is determined by a randomization via the relaxed control measure. The representations differ only in the realization of the randomization. The presence of the discontinuous indicator function in (3.5) does not affect the existence, uniqueness, or the approximation arguments, since it does not depend on the state. In fact, one could write the limit jump process as

$$\sum_i \int_0^t \int_\Gamma \int_0^1 I_{\{\gamma_0 \in (\mu_{i-1}(t),\, \mu_i(t)]\}} q(x(s-), \gamma, \alpha_i) \overline{N}_i(ds\, d\gamma\, d\gamma_0), \tag{3.6}$$

where the $\overline{N}_i(\cdot)$ are mutually independent and identically distributed. This representation in terms of a set of mutually independent Poisson measures works only if the control takes a finite or countable number of values.

Recapitulation. The above discussion suggests a generalization of the concept of Poisson measure which would allow the use of a continuum of control values within a well defined framework. In preparation for this, let $\{\mathscr{F}_t,\ t \ge 0\}$, $w(\cdot)$, $N(\cdot)$ be as in the Introduction. Let $u(\cdot)$ be an arbitrary admissible control with relaxed control representation $m(\cdot)$ and define the measure valued process $N_m(ds\, d\gamma\, d\alpha)$ as follows. Let $\Gamma_0 \in \mathscr{B}(\Gamma)$, and $U_0 \in \mathscr{B}(\mathscr{U})$. Then define $N_m([0, t] \times \Gamma_0 \times U_0) \equiv N_m(t, \Gamma_0, U_0)$ to be the number of jumps of $\int_0^t \int_\Gamma \gamma N(ds\, d\gamma)$ on $[0, t]$ with values in Γ_0, and where

$u(s) \in U_0$ at the jump times s. The stochastic model can then be written as

$$dx(t) = \int_{\mathscr{U}} b(x(t), \alpha) m_t(d\alpha)\, dt + \sigma(x(t))\, dw(t)$$
$$+ \int_{\mathscr{U}} \int_{\Gamma} q(x(t-), \gamma, \alpha) N_m(ds\, d\gamma\, d\alpha) + z(t), \qquad x(t) \in G. \tag{3.7}$$

The compensator of the counting measure valued process $N_m(\cdot)$ is the integral of

$$\lambda \Pi(d\gamma) m_t(d\alpha)\, dt \tag{3.8}$$

in the sense that the processes defined by

$$N_m(t, \Gamma_0, U_0) - \lambda \Pi(\Gamma_0) m(t, U_0)$$

are $\mathscr{F}_t$-martingales and are orthogonal for disjoint $\Gamma_0 \times U_0$. This is obvious from the facts that $I_{U_0}(u(\cdot))$ is progressively measurable,

$$\int_0^t \int_{\Gamma_0} I_{U_0}(u(s)) N(ds\, d\gamma) = N_m(t, \Gamma_0, U_0),$$

and the left hand side has compensator $\lambda \Pi(\Gamma_0) m(t, U_0)$ since $I_{U_0}(u(s)) = m_s(U_0)$. For bounded and measurable real-valued functions $\phi(\cdot)$,

$$\int_0^t \int_{\Gamma} \int_{\mathscr{U}} \phi(s, \gamma, \alpha) N_m(ds\, d\gamma\, d\alpha)$$
$$- \int_0^t \int_{\Gamma} \int_{\mathscr{U}} \phi(s, \gamma, \alpha) \lambda \Pi(d\gamma) m_s(d\alpha)\, ds \tag{3.9}$$

are also $\mathscr{F}_t$-martingales. Let $f(\cdot)$ be a bounded and continuous real valued function and define

$$p(t) = \int_0^t \int_{\Gamma} \int_{\mathscr{U}} \phi(s, \gamma, \alpha) N_m(ds\, d\gamma\, d\alpha).$$

Then the compensator for $f(p(\cdot))$ is

$$A(t) = \int_0^t \int_{\Gamma} \int_{\mathscr{U}} [f(p(s) + \phi(s, \gamma, \alpha)) - f(p(s))] \lambda \Pi(d\gamma) m_s(d\alpha)\, ds$$

in the sense that $f(p(t)) - f(p(0)) = A(t)$ plus an $\mathscr{F}_t$-martingale.

The set of integrands can be extended. The martingale property of (3.9) holds if any real valued, bounded $\mathscr{F}_t$-predictable process $\phi_0(\cdot)$ multiples $\phi(\cdot)$. Any left continuous and $\mathscr{F}_t$-adapted process is predictable. Note that, by its definition as the limit of the sequence of predictable processes in (2.1), $m_t(U_0)$ is predictable for any $U_0 \in \mathscr{B}(\mathscr{U})$.

The Relaxed Poisson Measure. The previous discussion exhibited the formulation, starting from the primitives $(w(\cdot), N(\cdot), m(\cdot))$. We are now in a position to develop the needed extension of the Poisson measure, which is consistent with the motivating discussion and (1.1).

Now, let us restart from the beginning. Let $\{\mathscr{F}_t, t \geq 0\}$ be a filtration and $w(\cdot)$ a standard $\mathscr{F}_t$-Wiener process and let $m(\cdot)$ be an admissible relaxed control. Let the (counting) measure valued process $N_m(\cdot)$ have the property that for any Borel sets Γ_0 and U_0, the processes

$$N_m(t, \Gamma_0, U_0) - \lambda \Pi(\Gamma_0) m(t, U_0) \tag{3.10}$$

are $\mathscr{F}_t$-martingales and are orthogonal for disjoint $\Gamma_0 \times U_0$. This martingale property and the fact that $m_t(\cdot)$ is $\mathscr{F}_t$-predictable constitutes the definition of *admissibility*. Such $N_m(\cdot)$ will be called *relaxed Poisson measures*. The martingale property and the fact that $N_m(\cdot)$ is a counting measure valued process specifies the distribution of $N_m(\cdot)$ uniquely. The appropriate weak topology is to be used on the space of measures, whatever the type.

Write the stochastic differential equation with controlled jumps in terms of the relaxed Poisson measure as

$$x(t) = x(0) + \int_0^t \int_{\mathscr{U}} b(x(s), \alpha) m_s(d\alpha)\, ds + \int_0^t \sigma(x(s))\, dw(s) + J(t) + z(t), \tag{3.11}$$

where $x(t) \in G$ and

$$J(t) = \int_0^t \int_\Gamma \int_{\mathscr{U}} q(x(s-), \gamma, \alpha) N_m(ds\, d\gamma d\alpha), \tag{3.12}$$

and $z(t) = \sum_i d_i y_i(t)$ is the reflection term. The $y_i(\cdot)$ are nondecreasing, can increase only at t where $x(t) \in \partial G_i$, and $y_i(0) = 0$.

For the motivating problem where the jumps were represented by either of (3.3), (3.4), or (3.6), the $N_{m^\delta}(ds\, d\gamma\, d\alpha)$ defined above (3.7) (for $m = m^\delta$) equals that defined here. Furthermore $N_{m^\delta}(\cdot)$ converges weakly to a relaxed Poisson measure $N_m(\cdot)$ associated with the limit relaxed control $m(\cdot)$. Also,

$$\int_0^t \int_\Gamma \int_{\mathscr{U}} q(x^\delta(s-), \gamma, \alpha) N_{m^\delta}(ds\, d\gamma d\alpha)$$

converges weakly to

$$\int_0^t \int_\Gamma \int_{\mathcal{U}} q(x(s-), \gamma, \alpha) N_m(ds\, d\gamma\, d\alpha).$$

In general, if $m_t(\cdot)$ is concentrated on a finite number of points $\{\alpha_i,\ i \le p\}$, then the jump processes can be represented in terms of mutually independent and identically distributed Poisson measures as in (3.6).

Implications of the Martingale Property. By the martingale property of (3.10), for bounded and measurable $\phi(\cdot)$

$$\int_0^t \int_\Gamma \int_{\mathcal{U}} \phi(s, \gamma, \alpha) N_m(ds\, d\gamma\, d\alpha) - \lambda \int_0^t \int_\Gamma \int_{\mathcal{U}} \phi(s, \gamma, \alpha) \Pi(d\gamma) m_s(d\alpha)\, ds$$

is an $\mathcal{F}_t$-martingale. This implies that the conditional (on $\mathcal{F}_t$) probability of a jump in any interval $[t, t+\delta)$ is $\lambda\delta + o(\delta)$. The probability of more than one jump is $o(\delta)$. It also tells us that the jump distribution is $\Pi(\cdot)$ and that the jump value (given a jump) is independent of the time of the jump. If $x(\cdot)$ is an $\mathcal{F}_t$-adapted process with paths in $D(\mathbb{R}^r; 0, \infty)$, then the process defined by

$$\int_0^t \int_\Gamma \int_{\mathcal{U}} q(x(s-), \gamma, \alpha) N_m(ds\, d\gamma\, d\alpha) - \lambda \int_0^t \int_\Gamma \int_{\mathcal{U}} q(x(s-), \gamma, \alpha) \Pi(d\gamma) m_s(d\alpha)\, ds \tag{3.13}$$

is an $\mathcal{F}_t$-martingale. The associated jump distribution is $q(x(s-), \gamma, \alpha)$ where γ is distributed as $\Pi(d\gamma)$ and α as $m_s(d\alpha)$ independently. Thus, the relaxed control plays the role of a randomization.

Next, let $\{\mathcal{F}_t^n,\ t \ge 0\}$, $w^n(\cdot), m^n(\cdot), N_{m^n}(\cdot)\}$ be a sequence of filtrations, standard $\mathcal{F}_t^n$-Wiener processes, admissible controls, and relaxed $\mathcal{F}_t^n$-Poisson measures. We have the following limit theorem.

THEOREM 3.1. *Under the conditions* (A2.1)–(A2.3), *the set*

$$\{x^n(\cdot), y^n(\cdot), w^n(\cdot), m^n(\cdot), N_{m^n}(\cdot)\}$$

is tight. The limit of any weakly convergent subsequence satisfies (3.11) *and* (3.12). *Let* $\{\mathcal{F}_t,\ t \ge 0\}$ *denote the filtration induced by the limit processes. Then* $w(\cdot)$ *is a standard* $\mathcal{F}_t$*-Wiener process,* $m(\cdot)$ *is admissible, and* $N_m(\cdot)$ *is an extended Poisson measure with compensator process defined by* (3.8).

Comments on the Proof. Only a few comments will be made. Since $\{EN_m(\cdot)\}$ is tight the set $\{N_m^n(\cdot)\}$ is also tight [12, Theorem 1.6.1]. The set $\{m^n(\cdot), w^n(\cdot)\}$ is always tight. The tightness of $\{y^n(\cdot)\}$ follows from the arguments in [15, Theorem 3.6.1] and the boundedness of G and the boundary condition (A2.3). Then the tightness of $\{x^n(\cdot)\}$ follows from standard weak convergence arguments. Suppose that (abusing terminology) n indexes a weakly convergent subsequence, with limit denoted by $(x(\cdot), y(\cdot), w(\cdot), m(\cdot), N_m(\cdot))$ and let $\{\mathscr{F}_t,\ t \geq 0\}$ denote the filtration engendered by the limit process. The nonanticipativity, the Wiener and martingale properties, and the admissibility of $m(\cdot)$ follow by standard weak convergence arguments. Note, in particular, that $N_m(\cdot)$ is a relaxed $\mathscr{F}_t$-Poisson measure associated with $m(\cdot)$. Since $q(\cdot)$ is continuous and bounded,

$$J^n(t) = \int_0^t \int_\Gamma \int_{\mathscr{U}} q(x^n(s-), \gamma, \alpha) N_{m^n}(ds\, d\gamma\, d\alpha)$$

converges weakly to $J(\cdot)$. Now, piece the interjump limits and jump limits together to get (3.11) and (3.12).

Existence of an Optimal Control. The weak sense uniqueness implies that the jump terms and control can be approximated and that there is an optimal control. Define $V_\beta(x) = \inf_m W(x, m)$, where the inf is over the relaxed admissible controls and the system is (3.11) and (3.12). The weak convergence in Theorem 3.1 and the fact that the $m^n(\cdot)$ can be chosen to be $1/n$-optimal controls implies the following theorem.

THEOREM 3.2. *Assume* (A2.1)–(A2.3). *Then there is an optimal control of the relaxed problem.*

Comment on the Proof. Let $(x^n(\cdot), y^n(\cdot), m^n(\cdot), w^n(\cdot), N_{m^n}(\cdot))$ denote a minimizing sequence and, abusing terminology, let n index a weakly convergent subsequence with limit $(x(\cdot), y(\cdot), m(\cdot), w(\cdot), N_m(\cdot))$. It follows from the proof of [16, Theorem 11.1.1] that

$$\sup_n E|y^n(t)|^2 = O(t). \tag{3.14}$$

This, the boundedness of $k(\cdot)$ on G, and the weak convergence imply that

$$W_\beta(x, m^n) \to W_\beta(x, m) = \inf_m W_\beta(x, m) \equiv V_\beta(x) \tag{3.15}$$

which is the theorem.

Approximating the Optimal Relaxed Control and Relaxed Poisson Measure. Let $\rho > 0$ and divide $\mathscr{U}$ into a finite number of disjoint connected subsets U_i^ρ, $i \leq p_\rho$, with diameters less than ρ and let α_i^ρ be a point in U_i^ρ. Given an admissible relaxed control $m(\cdot)$, define $m^{\rho}(\cdot)$ by $m^{\rho}(t, \alpha_i^{\rho}) = m(t, U_i^{\rho})$, $i \leq p_\rho$. Thus, $m(\cdot)$ is approximated by an admissible relaxed control with values in a finite set. The measure $N_{m^\rho}(\cdot)$ is constructed from $N_m(\cdot)$ in the obvious way.

The following theorems are used to extend the approximation results in [16, Subsect. 10.1.2 and Sect. 10.3] to the problem with controlled jumps.

THEOREM 3.3. *Assume* (A2.1)–(A2.3) *and let* $\rho > 0$. *Let*

$$\left(x^{\rho}(\cdot), y^{\rho}(\cdot), m^{\rho}(\cdot), w^{\rho}(\cdot), N_{m^{\rho}}(\cdot)\right)$$

solving (3.11) *and* (3.12) *be given. Then the set converges weakly to the solution of* (3.11), (3.12), *and* $W_\beta(x, m^{\rho}) \to W_\beta(x, m)$.

THEOREM 3.4. *Assume* (A2.1)–(A2.3) *and that* $\mathcal{U}$ *has only finitely many points* $\{\alpha_i,\ i \leq p\}$. *Let* $(w(\cdot), N(\cdot), m(\cdot))$ *be admissible. Define the piecewise constant control* $u^{\Delta}(\cdot)$ *as follows. Define* $\tau_i^{\Delta}(l) = m(l\Delta, \alpha_i) - m(l\Delta - \Delta, \alpha_i)$ *and divide each* $[l\Delta, l\Delta + \Delta)$ *into subintervals of lengths* $\tau_1^{\Delta}(l), \dots, \tau_p^{\Delta}(l)$. *Then use the control value* α_i, $i \leq p$, *on the subintervals successively. Let* $m^{\Delta}(\cdot)$ *denote the relaxed control representation of* $u^{\Delta}(\cdot)$. *Let* $N_{m^{\Delta}}(\cdot)$ *denote the associated relaxed Poisson measure, and* $(x^{\Delta}(\cdot), y^{\Delta}(\cdot))$ *the solution to* (3.12) *and* (3.13). *Then*

$$\left(x^{\Delta}(\cdot), y^{\Delta}(\cdot), w^{\Delta}(\cdot), m^{\Delta}(\cdot), N_{m^{\Delta}}(\cdot)\right)$$

converges weakly to $(x(\cdot), y(\cdot), w(\cdot), m(\cdot), N_m(\cdot)))$, *solving* (3.11) *and* (3.12).

Infima over Ordinary Controls. Theorems 3.3 and 3.4 imply that the infimum of the costs over the ordinary admissible controls equals the infimum over the relaxed controls. Thus, the extension of the model via the introduction of the relaxed Poisson measure does not affect the infimum of the cost function.

Representation by a Standard Poisson Measure. Let $u(\cdot)$ be admissible, piecewise constant, and take only finitely many values $\{\alpha_i,\ i \leq p\}$, as in Theorem 3.4. Then the jump term can be represented in terms of a standard Poisson measure. Let $I_i(t)$ be the indicator function of the event that $u(t) = \alpha_i$, which we can take to be a predictable process. Then the jump term is

$$\sum_i \int_0^t \int_\Gamma I_i(s) q(x(s-), \gamma, \alpha_i) N(ds\, d\gamma), \tag{3.16}$$

for a standard Poisson measure with jump rate λ and jump distribution $\Pi(\cdot)$.

4. THE BELLMAN EQUATION FOR THE POLLING PROBLEM

The stochastic differential equation for the polling problem with service interruptions can be written as [1]

$$dx = b\,dt + dw + dJ + dz, \qquad x \in [0, B],\ B < \infty, \tag{4.1}$$

where $z(\cdot)$ is the reflection term at the end points $0, B$. The control u is the workload in queue 1, and $x - u$ is the workload in queue 2. The cost rate is $k(\cdot)$ and $c = 0$. The distributions of the jumps in $x(\cdot)$ are defined by (1.3) and the discussion above it. The parameter b represents the scaled difference between the speed needed to handle the average requirements and the actual server speed, and it is usually negative, which gives the system some excess capacity. The Wiener process $w(\cdot)$ represents the randomness in the work arrival process.

The contribution of the jump term to the differential generator, acting on bounded and measurable functions, is

$$\lambda \int_{\mathscr{U}} \int_{\Gamma} [f(x + q(x, \alpha, \gamma)) - f(x)] \Pi(d\gamma) m_t(d\alpha). \tag{4.2}$$

Here, once again, the relaxed control $m(\cdot)$ plays the role of a randomization. The distribution of the value of a jump at t is the distribution of $q(x(t-), \alpha, \gamma))$, where (γ, α) are distributed according to $\Pi(\cdot) \times m_t(\cdot)$.

This is a one-dimensional problem with $G = \{x : 0 \le x \le B\}$. The upper bound B might not appear in the original problem, but is inserted here since the state space needs to be bounded for numerical purposes. Let $\mathscr{L}$ denote the differential generator of the pure diffusion part of (4.1). Write the control in feedback form $u(x(t-))$. Then (4.2) can be written as

$$\sum_i \bar{\lambda}_i^s E[f(x + \xi_i^v) - f(x)],$$

where ξ_i^v is the jump due to a vacation of source i, and E denotes the expectation of the jump given the values of the state and the control at the jump.

Define the function

$$H(V_\beta, x) = \min_{u(x) \le x} \left\{ k(x, u(x)) + \sum_i \bar{\lambda}_i^s E\big[V_\beta(x + \xi_i^v) - V_\beta(x)\big] \right\}. \tag{4.3}$$

The formal Bellman equation is the partial differential integral equation

$$\mathscr{L} V_\beta(x) - \beta V_\beta(x) + H(V_\beta, x) = 0, \tag{4.4}$$

with the boundary condition $V_{\beta, x}(0) = 0$, where the subscript x denotes the derivative.

The existence of an optimal control was established in Theorem 3.2.

5. THE NUMERICAL ALGORITHM

Consider the numerical problem of computing $V_\beta(x)$ with the system (1.1) or (3.11), (3.12). The details of the construction of the class of effective algorithms known as the Markov chain approximation method is in [16, Sect. 5.6]. The algorithm constructs an "approximating" discrete time controlled Markov chain which is parameterized by a discretization parameter h. The approximation procedure provides the chain and an "interpolation interval" $\Delta^h(x, u)$. When suitably interpolated into a continuous time process with the (possibly) state and control dependent interpolation intervals $\Delta^h(x, u)$, the sequence converges weakly to a controlled reflected jump-diffusion. The conditions are minimal. Apart from (A2.1)–(A2.3), the main condition for the approximating chains is what is known as "local consistency," which essentially means that the "local" conditional means and variances of the one step transitions are "close" to those for the reflected jump-diffusion over the same time interval. Convergence proofs are given in [16] for all of the usual cost functions. The solution of the numerically feasible optimal control problem for the approximating chain approximates the optimal cost function for the original problem. Also, approximations to the optimal control are obtained.

The convergence proofs in [16] do not cover the case of controlled jumps, but the methods in [16] are readily adapted using the ideas presented above. Only an outline will be given. The construction of the algorithms when the jump is controlled differs little from that in [16, Sect. 5.6]. One just adds a control to the transition probability for the jumps. The approximating chain is interpolated into a continuous parameter process $\psi^h(\cdot)$ which can be represented in the form

$$\psi^h(t) = \psi^h(0) + \int_0^t b\big(\psi^h(s), u^h(s)\big)\, ds + \int_0^t \sigma\big(\psi^h(s)\big)\, dw^h(s) + J^h(t) + z^h(t) + \epsilon^h(t), \quad (5.1)$$

where $u^h(\cdot)$ is the control process, $\epsilon^h(\cdot)$ converges weakly to the "zero" process, and $z^h(\cdot) = \sum_i d_i y_i^h(\cdot)$ is the reflection term, where $y_i^h(\cdot)$ can increase only at t where $\psi^h(t) \in \partial G_i$. The process $\psi^h(\cdot)$ is actually a continuous time controlled Markov chain on a finite state space (a discretization of G). Given the current state x and control u, the mean sojourn time at x is $\Delta t^h(x,u)$. The process $w^h(\cdot)$ converges weakly to a standard Wiener process and $J^h(\cdot)$ is the jump term, which can be represented as

$$J^h(t) = \int_0^t \int_{\Gamma^h} \int_{\mathcal{U}} q(\psi^h(s), \gamma, u^h(s)) N^h(ds\, d\gamma), \tag{5.2}$$

where $N^h(\cdot)$ is a "proto Poisson measure." Although it is confined to having jumps only at the end of the interpolation intervals, it has the "effective" jump rate λ and jump distribution $\Pi^h(\cdot)$, an approximation which converges weakly to $\Pi(\cdot)$ as $h \to 0$. The formulation does not use relaxed controls at this point, since the controls which are used in (5.1) and (5.2) are obtained from the solution of the Bellman equation for the discrete approximation and so will be ordinary controls, which are constant on the interpolation intervals. As $h \to 0$, the pair $w^h(\cdot), N^h(\cdot)$ converges weakly to the Wiener process and Poisson measure of Section 2. Let $V_\beta^h(x)$ denote the infimum of the costs for the numerical approximation.

To prove the convergence $V_\beta^h(x) \to V_\beta(x)$, one needs to start by proving the weak convergence of $\psi^h(\cdot)$ to a controlled diffusion. Then convergence of the costs can be done. We will comment on the first step only. The second step follows the lines used in [16], with the relaxed Poisson measure used. The idea is to show that the weak sense limit satisfies (3.11) and (3.12). Rewrite (5.1) in terms of the relaxed control representations

$$\begin{aligned} \psi^h(t) = \psi^h(0) &+ \int_0^t \int_{\mathcal{U}} b(\psi^h(s), \alpha) m_s^h(d\alpha)\, ds + \int_0^t \sigma(\psi^h(s))\, dw^h(s) \\ &+ J^h(t) + z^h(t) + \epsilon^h(t), \end{aligned} \tag{5.3}$$

$$J^h(t) = \int_0^t \int_{\Gamma^h} \int_{\mathcal{U}} q(\psi^h(s), \gamma, \alpha) N_{m^h}^h(ds\, d\gamma\, d\alpha). \tag{5.4}$$

From this point on, the convergence proof is what is used for Theorem 3.1 in showing that $(\psi^h(\cdot), y^h(\cdot), m^h(\cdot), w^h(\cdot), N_{m^h}^h(\cdot))$ converges weakly to $(\psi(\cdot), y(\cdot), m(\cdot), w(\cdot), N_m(\cdot))$.

An Approximation of the Optimal Control. Let $W_\beta^h(x, m^h)$ denote the cost for the numerical approximation when $m^h(\cdot)$ is the relaxed control representation of the actual control which is used. In [16, Chap. 10], the proof that $V_\beta^h(x) \to V_\beta(x)$ required the use of a convenient (for mathematical purposes only) approximation to the optimal control for (3.11) and (3.12). The following theorem is a restatement of that result for the present case. It exploits the fact that $x(\cdot)$ can be represented in terms of a standard Poisson measure when the control takes only a finite number of values and is piecewise constant.

THEOREM 5.1 [16, a slight modification of Theorem 3.1, Chap. 10, with the same proof]. *Assume* (A2.1)–(A2.3). *Fix* $\epsilon > 0$, *and let* $(x(\cdot), z(\cdot), m(\cdot), w(\cdot), N_m(\cdot))$ *be an* ϵ*-optimal solution to* (3.11) *and* (3.12). *Then, there is a* $\delta > 0$ *and a probability space on which are defined a pair* $(w^{\epsilon}(\cdot), N^{\epsilon}(\cdot))$, *an admissible control* $u^{\epsilon}(\cdot)$ *which takes values in a finite set* $\{\alpha_1^{\epsilon}, \dots, \alpha_{p_\epsilon}^{\epsilon}\} = \mathscr{U}_\epsilon \subset \mathscr{U}$, *and is constant on the intervals* $[n\delta, n\delta + \delta)$, *and a solution* $x^{\epsilon}(\cdot)$ *such that*

$$\left| W_\beta(x, m^\epsilon) - W_\beta(x, m) \right| \leq \epsilon. \tag{5.5}$$

There is $\theta > 0$ *and a partition* $\{\Gamma_j^q, j \leq q\}$ *of* Γ *such that* $\Pi(\partial \Gamma_j^q) = 0$, *all* j, *and the approximating* $u^{\epsilon}(\cdot)$ *can be chosen so that its probability law at any time* $k\delta$, *conditioned on* $\{x, w^{\epsilon}(s), N^{\epsilon}(s),\ s \leq k,\ u^{\epsilon}(i\delta),\ i < k\}$, *depends only on the initial condition* x *and on*

$$\left\{ w^\epsilon(p\theta), N^\epsilon\left(p\theta, \Gamma_j^q\right), j \leq q, p\theta < k\delta; u^\epsilon(i\delta), i < k \right\} \tag{5.6}$$

and is continuous in the $w_\epsilon(p\theta)$ *and* x *arguments for each value of the other arguments. More particularly, there are functions* $q_k(\cdot)$ *which are continuous in the* w *and* x *variables for each value of the other variables and such that*

$$\begin{aligned} &P\{u^\epsilon(k\delta) = \alpha \,|\, x, u^\epsilon(i\delta), i < k, w(s), N(s), s \leq k\delta\} \\ &\quad = q_k\left(\alpha; x, u^\epsilon(i\delta), i < k, w^\epsilon(p\theta), N^\epsilon\left(p\theta, \Gamma_j^q\right), j \leq q, p\theta < k\delta\right). \end{aligned} \tag{5.7}$$

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